$$= m \log \left[\frac{(x^2-y^2)(x^2-y^2-c^2)+4x^2y^2}{(x^2-y^2-c^2)^2+4x^2y^2} + \frac{i2xy\,(\cancel{x^2}-\cancel{y^2}-c^2-\cancel{x^2}+\cancel{y^2})}{(x^2-y^2-c^2)^2+4x^2y^2} \right]$$

$$= m \log \left(\frac{(x^2-y^2)(x^2-y^2-c^2)+4x^2y^2}{(x^2-y^2-c^2)^2+4x^2y^2} + \frac{i(-2xy\,c^2)}{(x^2-y^2-c^2)^2+4x^2y^2} \right)$$

$$w = m \log (x+iy)$$

$$\left[\because \log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1} y/x\right]$$

$$\psi = \quad Im(w) = m \tan^{-1} y/x.$$

$$= m \tan^{-1} \left[\left(\frac{-2xyc^2}{(x^2-y^2-c^2)^2+4x^2y^2} \right) \times \right.$$

$$\left.\left(\frac{(x^2-y^2-c^2)^2+4x^2y^2}{(x^2-y^2)(x^2-y^2-c^2)+4x^2y^2}\right)\right]$$

$$= m\tan^{-1}\left(\frac{(-2xyc^2)}{(x^2-y^2)(x^2-y^2-c^2)+4x^2y^2}\right)$$

$$= m\tan^{-1}\left(\frac{-2xyc^2}{(x^2-y^2)^2-c^2(x^2-y^2)+4x^2y^2}\right)$$

$$= m\tan^{-1}\left(\frac{-2xyc^2}{(x^4+y^4-2x^2y^2+4x^2y^2)-c^2(x^2-y^2)}\right)$$

$$\psi = m\tan^{-1}\left(\frac{-2xyc^2}{(x^4+y^4+2x^2y^2)-c^2(x^2-y^2)}\right)$$

$$\psi = m\tan^{-1}\left(\frac{-2xyc^2}{(x^2+y^2)^2-c^2(x^2-y^2)}\right)$$

$$k = \frac{-2xyc^2}{(x^2+y^2)^2-c^2(x^2-y^2)}$$

$$\Rightarrow -2xyc^2 = K(x^2+y^2)^2 - c^2K(x^2-y^2)$$

$$\Rightarrow K(x^2+y^2)^2 = c^2(K(x^2-y^2)) - 2xy)$$

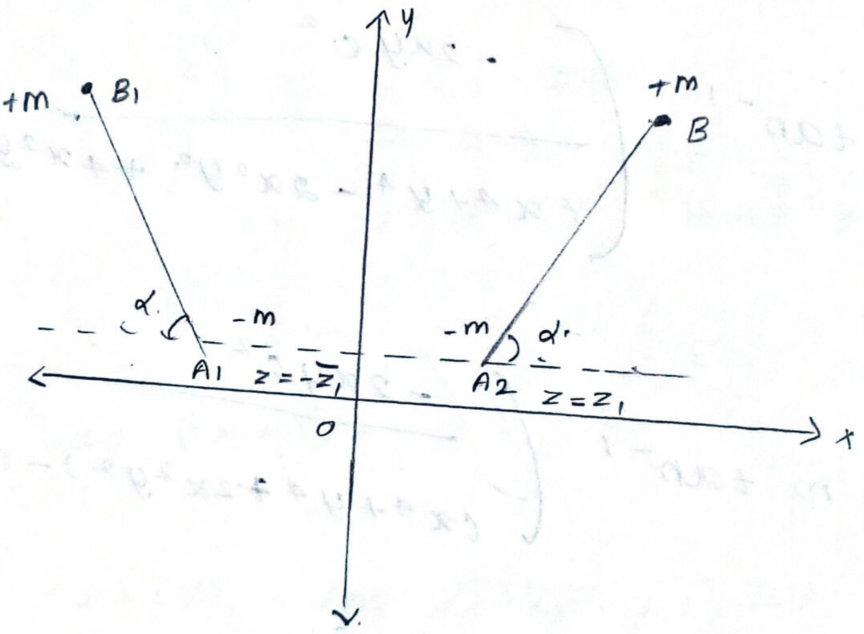
$$\cancel{K}(x^2+y^2)^2 = \cancel{K}c^2\left[(x^2-y^2) - \frac{2}{K}xy\right]$$

$$(x^2+y^2)^2 = c^2(x^2-y^2) - K xy$$

where,

$$K = \text{constant} = 2/K.$$

TWO-DIMENSIONAL IMAGE SYSTEM :-



we first show that the image of a line source in a rigid infinite plane is a line source of equal strength at the optimal image in the plane of the point at which the given line source is situated.

suppose the plane is $x=0$ and that there is a line source of strength m at $z=z_1$.

where,

$Re(z_1) \geq 0$.

If we now remove the boundary and introduce an equal line source at $z=-\bar{z}_1$, the mirror image of z_1 in $x=0$, then the complex velocity potential at P, where $\vec{OP}=z$ is,

$$W = -m \log (z-z_1) - m \log \underbrace{(z+\bar{z}_1)}_{[z-(-\bar{z}_1)]}$$

$$\frac{dw}{dz} = -m(z-z_1)^{-1} - m(z+\bar{z}_1)^{-1}.$$

At $z=iy$ on the plane $x=0$

$$\therefore \left.\frac{dw}{dz}\right]_{z=iy} = -m[iy-z_1]^{-1} - m[iy+\bar{z}_1]^{-1}.$$

$$\frac{dw}{dz} = -m\left[(iy-z_1)^{-1} + (iy+\bar{z}_1)^{-1}\right]$$

$$\frac{dw}{dz} = m\{(z_1 - iy)^{-1} - (\bar{z}_1 + iy)^{-1}\}$$

since,

$(z_1 + iy)^{-1}$ is the complex conjugate of $(z_1 - iy)^{-1}$.

$$\frac{dw}{dz} = m\{(z_1 - iy)^{-1} - \overline{(z_1 - iy)^{-1}}\}$$

$$\Rightarrow Re\left(\frac{dw}{dz}\right) = 0.$$

This shows that on the plane $x = 0$, the velocity component $u = 0$, and hence no fluid crosses this plane and hence we can replaced by a rigid barrier.

To find the image of a two-dimensional doublet in a rigid plane oy.

From the figure in which there are line sources at the points A and B.

taken very close together of strengths $-m$ and $+m$ per unit length.

This represents respective images in OY are $-m$ at A', m at B'. where A', B' are the reflection of A, B in OY.

The line $\overline{AB}$ makes angle α with OX.

Thus, $\overline{A'B'}$ makes angle $(\pi-\alpha)$ with $\overline{OX}$.

In the limit as $m \to \infty$, $AB \to 0$.

we have equal line doublets at A, B with their axes inclined at α $(\pi-\alpha)$ to $\overline{OX}$.

To determine the image of a line vortex in an infinite plane.

Consider a <u>vortex</u> in a infinite plane of strengths K at $z=z_1$.

where,

$Re(z_1)>0$.

Introduce a line vortex of strength $-K$ at $z=-\overline{z_1}$ the reflection of z_1 in the plane $z = 0$.

Then the complex potential at z is,

$$W = \left(\frac{iK}{2\pi}\right) \{ \log (z - z_1) - \log (z + \bar{z}_1) \}$$

so that,

$$\frac{dw}{dz} = \left(\frac{iK}{2\pi}\right) \{ (z - z_1)^{-1} - (z + z_1)^{-1} \}$$

on the plane $x = 0, z = iy$.

$$\frac{dw}{dz} = \frac{iK}{2\pi} \{ (iy - z_1)^{-1} - (iy + \bar{z}_1)^{-1} \}$$

$$= \frac{K}{2\pi} \{ i (iy - z_1)^{-1} - i (iy + \bar{z}_1)^{-1} \}$$

$$= \frac{K}{2\pi} \{ -i (z_1 - iy)^{-1} - i (\bar{z}_1 + iy)^{-1} \}$$

since, $-i (\bar{z}_1 + iy)^{-1} \cdot i (\bar{z}_1 - iy)^{-1}$ are complex conjugates.

$$Re \left(\frac{dw}{dz}\right) = 0 \text{ on } x = 0.$$

showing that no fluid across the plane.

Hence a rigid plane infinite surface may be introduced into the position $x=0$ and the appropriate image of a line vortex in such a plane is an equal and opposite line vortex at the mirror image in the plane.

PROBLEM:-

1. Describe the irrotational motion of an incompressible liquid for which the complex potential is, $w = ik\log z$

two parallel line vertices of strength K_1, K_2. $(K_1+K_2 \neq 0)$ in unlimited liquid cross the z-plane at points A, B respectively. The centre of masses K_1 at A and K_2 at B is, G.

Show that, if the motion of the liquid is due to solely to this vertices, G is a fixed point about which A, B move in circles with angular velocity $\frac{K_1+K_2}{(AB)^2}$. Show also that the fluid speed at any point P in the z plane

is,

$$\frac{(K_1 + K_2) CP}{AP \cdot BP}, \text{ where } C \text{ is the centre of}$$

the masses K_2 at A, K_1 at B.

PROOF:-

$$w = ik \log z \quad \text{—} (1)$$

Replacing $z = re^{i\theta}$ and seperating the

real and imaginary part,

$$\phi = -K\theta$$

$$\psi = K \log r.$$

The stream lines are given by ψ = constant.

which are concentric circles centred at O.

The equipotential are given by ϕ = constant

which are radial vector through O cutting

the circles orthogonally.

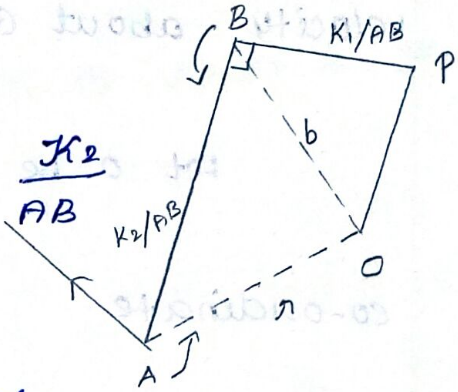
Also at (r, θ) in the plane of flow the velocity components are,

$$q_r = 0$$

$$q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{K}{r} \quad \Bigg\} \rightarrow (2)$$

Figure shows the line vertices K_1 at A and K_2 at B. by (2) K_1 at A gives the velocity $\frac{K_1}{AB}$.

Illy,



K_2 at B gives the velocity $\frac{K_2}{AB}$ as shown in the figure

Associating the masses K_1 at A K_2 at B we see that the total linear momentum of the system in the direction of the velocity of B is,

$$K_1\left(-K_2/AB\right) + K_2\left(\frac{K_1}{AB}\right) = 0.$$

Hence the centroid G of the two masses does not move.

The relative velocity of B with respect to A is velocity B − velocity A.

$$\Rightarrow \frac{K_2}{AB} - \left(\frac{-K_1}{AB}\right)$$

$$\Rightarrow \frac{K_1 + K_2}{(AB)^2}$$

From ②,

The angular velocity of AB is $\frac{K_1 + K_2}{(AB)^2}$

Hence the line rotates with this angular velocity about G.

Let O be the fixed origin on the co-ordinate.

suppose that $\overline{OA} = a$, $\overline{OB} = b$, a and b being complex numbers.

Let P be a point for which $\overline{OP} = z$.

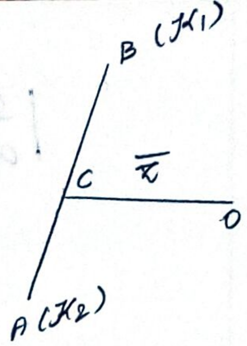
Then, $\overline{AP} = z - a$

$\overline{BP} = z - b$

The complex potential at P due to A and B is,

$$w = i K_1 \log (z-a) + i K_2 \log (z-b).$$

$$\frac{dw}{dz} = \frac{iK_1}{z-a} + i\frac{K_2}{z-b}$$

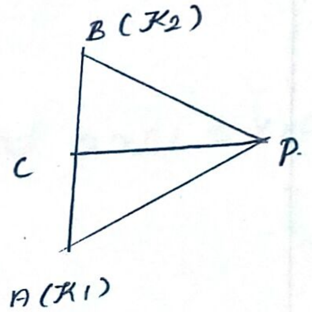


$$= \frac{iK_1(z-b) + iK_2(z-a)}{(z-a)(z-b)}$$

$$\frac{dw}{dz} = i\left[\frac{K_1(z-b)+K_2(z-a)}{(z-a)(z-b)}\right] \rightarrow (*)$$

If c denote the centroid of K_2 at A, K_1 at B and if $OC = \bar{z}$ then,

$$\overline{CP} = \frac{K_2\,\overline{AP} + K_1\,\overline{BP}}{K_1+K_2}.$$



$$(K_1+K_2)\,\overline{CP} = K_2\overline{AP} + K_1.\overline{BP}$$

$$(K_1+K_2)\,\overline{CP} = K_2(z-a) + K_1(z-b)$$

$$(K_1+K_2)(z-\bar{z}) = K_2(z-a) + K_1(z-b).$$

Apply ④ in ③,

$$\frac{dw}{dz} = i\left[\frac{(K_1+K_2)(z-\bar{z})}{(z-a)(z-b)}\right]$$

$$\left|\frac{dw}{dz}\right| = \frac{(K_1+K_2)(z-\bar{z})}{(z-a)(z-b)}$$

$$\left|\frac{dw}{dz}\right| = \frac{(K_1 + K_2)\,CP}{AP \cdot BP}$$

Hence the proof